

THETA DIVISORS OF ABELIAN VARIETIES AND PUSH-FORWARD HOMOMORPHISM AT THE LEVEL OF CHOW GROUPS

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ABSTRACT. In this text we prove that if an abelian variety A admits of an embedding into the Jacobian of a smooth projective curve C , and if we consider Θ_A to be the divisor $\Theta_C \cap A$, where Θ_C denotes the theta divisor of $J(C)$, then the embedding of Θ_A into A induces an injective push-forward homomorphism at the level of Chow groups. We show that this is the case for every principally polarized abelian varieties. We further prove that the above result can be obtained for a family of abelian varieties embedded into a family of Jacobians.

1. INTRODUCTION

In the paper [BI] the authors were discussing the following. Let C be a smooth projective curve of genus g and let Θ denote the theta divisor embedded into the Jacobian $J(C)$ of the curve C . Let j denote this embedding. Then the push-forward homomorphism j_* at the level of Chow groups is injective. Also in this paper the author discussed about the push-forward homomorphism at the level of Chow groups induced by the closed embedding of some special divisors in the Jacobian $J(C)$, arising from finite, étale coverings of the curve C , see [BI][theorem 4.1]. So we thought it would be worth discussing the same problem for an arbitrary abelian variety A . That is let A be an abelian variety and let H denote a divisor embedded inside A . Let j denote this embedding. Then can we say that the push-forward homomorphism j_* at the level of Chow groups is injective? This question is affirmatively answered in the case when the abelian variety A is embedded in the Jacobian variety and we consider the divisor $\Theta \cap A$ inside A , where Θ is the theta divisor of $J(C)$. This is exactly the case of Prym-Tyurin

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varieties, which are abelian varieties embedded inside some Jacobian variety, and the intersection of the theta divisor of $J(C)$ with A is linearly equivalent to some multiple of the Theta divisor of A . Since any principally polarized abelian variety is a Prym-Tyurin variety of some exponent (see [BL] corollary 12.2.4), so for the principally polarized abelian varieties the above question about the injectivity of the push-forward homomorphism at the level of Chow groups is partially answered, when the divisor H is the intersection of the abelian variety A with the theta divisor of the ambient Jacobian variety, where A is embedded.

Let A be a principally polarized abelian variety embedded into $J(C)$ for some smooth projective curve C . Let Θ be the theta divisor of $J(C)$. Then the embedding of $\Theta \cap A$ into A induces an injective push-forward homomorphism at the level of Chow groups

In the last section we do everything for a family of abelian varieties \mathcal{A} , which is embedded in some family of Jacobians \mathcal{J} . Let Θ denote the family of theta divisors of the Jacobians in the family \mathcal{J} and consider $\Theta \cap \mathcal{A}$. Then we prove that the embedding of $\Theta \cap \mathcal{A}$ into A induces an injective push-forward homomorphism at the level of Chow groups. This in particular, implies the following. Take a family of principally polarized abelian varieties \mathcal{A} , then by the corollary 12.2.4 in [BL], there exists a family of Jacobian varieties \mathcal{J} in which the given family of abelian varieties \mathcal{A} can be embedded. Then consider Θ to be the family of theta divisors of \mathcal{J} and $i : \Theta \cap \mathcal{A} \rightarrow \mathcal{A}$. Then it follows that i_* is an injection at the level of Chow groups.

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2. ABELIAN VARIETIES EMBEDDED IN JACOBIANS

Let A be an abelian variety embedded inside the Jacobian of a smooth projective curve C . Let Θ_C denote the theta divisor of the Jacobian $J(C)$. Consider Θ_A to be $\Theta_C \cap A$, and the closed embedding of Θ_A into A , denote it by j_A , then we prove that j_{A*} is injective from $CH_*(\Theta_A)$ to $CH_*(A)$. First we mention the following theorem which will be used in this text.

Theorem 2.1. *The homomorphism j_{A*} is injective at the level of Chow groups.*

Proof. To prove this theorem we use the localisation exact sequence for Higher Chow groups as present in [BI]. First let us consider inverse image of A, Θ_A in $\text{Sym}^g C, \text{Sym}^{g-1} C$ respectively, denote them by A', Θ'_A . Then we have the following commutative diagram with the rows exact.

$$\begin{array}{ccccccc}
CH_k(\text{Sym}^{g-1} C, 1) & \longrightarrow & CH_k(\text{Sym}^{g-1} C \setminus \Theta'_A, 1) & \longrightarrow & CH_k(\Theta'_A) & \longrightarrow & CH_k(\text{Sym}^{g-1} C) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
CH_k(\text{Sym}^g C, 1) & \longrightarrow & CH_k(\text{Sym}^g C \setminus A', 1) & \longrightarrow & CH_k(A') & \longrightarrow & CH_k(\text{Sym}^g C)
\end{array}$$

Now by the higher dimensional analog of Collino's theorem as proved in [BI] we have the homomorphism

$$CH_k(\text{Sym}^{g-1} C, 1) \rightarrow CH_k(\text{Sym}^g C, 1)$$

is injective and also

$$CH_k(\text{Sym}^{g-1} C \setminus \Theta'_A, 1) \rightarrow CH_k(\text{Sym}^g C \setminus A')$$

is injective. For proofs of this please see [BI] section 5. Also by the Collino's theorem about Chow groups of symmetric powers [Co] we have $CH_k(\text{Sym}^{g-1} C) \rightarrow CH_k(\text{Sym}^g C)$ injective. Now we want to prove that $CH_k(\Theta'_A) \rightarrow CH_k(A')$ is injective. So suppose we take some non-zero element in $CH_k(\Theta'_A)$. Suppose it is mapped to a non-zero element in $CH_k(\text{Sym}^{g-1} C)$, then since $CH_k(\text{Sym}^{g-1} C) \rightarrow CH_k(\text{Sym}^g C)$ is injective, we have that the non-zero element is further mapped to a non-zero element in $CH_k(\text{Sym}^g C)$. Therefore by the commutativity of the square

$$\begin{array}{ccc}
CH_k(\Theta'_A) & \longrightarrow & CH_k(\text{Sym}^{g-1} C) \\
\downarrow & & \downarrow \\
CH_k(A') & \longrightarrow & CH_k(\text{Sym}^g C)
\end{array}$$

we get that the element we started with is mapped to a non-zero element in $CH_k(A')$, under the homomorphism $CH_k(\Theta'_A) \rightarrow CH_k(A')$. Suppose that the element is mapped to zero in $CH_k(\text{Sym}^{g-1}C)$. Then by the exactness of the first row we get that it is in the image of the homomorphism $CH_*(\text{Sym}^{g-1}C \setminus \Theta'_A, 1) \rightarrow CH_k(\Theta'_A)$. Since the element we started with was non-zero, its pre-image in $CH_*(\text{Sym}^{g-1}C \setminus \Theta'_A, 1)$ is non-zero. The homomorphism $CH_*(\text{Sym}^{g-1}C \setminus \Theta'_A, 1) \rightarrow CH_*(\text{Sym}^gC \setminus A', 1)$ is injective. So the pre-images of the element chosen are mapped to non-zero elements in $CH_*(\text{Sym}^gC \setminus A')$. Now these elements can be mapped to zero in $CH_k(A')$. If they are mapped to a non-zero element, we get that the element we started with is mapped to a non-zero element. If not we have the pre-images of the elements are mapped inside the image of the homomorphism

$$CH_*(\text{Sym}^gC, 1) \rightarrow CH_*(\text{Sym}^gC \setminus A', 1) .$$

Since there exists elements in $CH_*(\text{Sym}^gC, 1)$ which are mapped mapped to the images of the pre-images under the homomorphism

$$CH_*(\text{Sym}^{g-1}C \setminus \Theta'_A, 1) \rightarrow CH_*(\text{Sym}^gC \setminus A', 1) .$$

Since these elements are supported on $\text{Sym}^{g-1}C \setminus \Theta'_A$, we get that the element in $CH_k(\text{Sym}^gC)$ is in the image of the homomorphism $CH_k(\text{Sym}^{g-1}C) \rightarrow CH_k(\text{Sym}^gC)$. Therefore we get an element in $CH_k(\text{Sym}^{g-1}C)$ which is mapped to the element that we started with in $CH_k(\Theta'_A)$. Then by the localisation exact sequence it will follow that the element we started with is zero contradicting our assumption that it is non-zero. So the pre-images of the element we started with are mapped to a non-zero element in $CH_k(A')$. So the push-forward homomorphism from $CH_k(\Theta'_A)$ to $CH_k(A')$ is injective. Now consider the following commutative diagram.

$$\begin{array}{ccc} CH_k(\Theta'_A) & \longrightarrow & CH_k(A') \\ \downarrow & & \downarrow \\ CH_k(\Theta_A) & \longrightarrow & CH_k(A) \end{array}$$

We have $\Theta'_A \rightarrow \Theta_A$ surjective and birational and $A' \rightarrow A$ surjective and birational. Since Sym^gC is a projective bundle over $J(C)$, we

have Θ'_A, A' are projective bundles over Θ_A, A . So we get a section of $\Theta'_A \rightarrow \Theta_A$ and $A' \rightarrow A$ onto some open subsets of Θ'_A, A' respectively. Now we can apply theorem [BI][theorem 3.1] to deduce that the push-forward homomorphism from $CH_k(\Theta_A)_{\mathbb{Q}}$ to $CH_k(A)_{\mathbb{Q}}$ is injective. \square

Example 2.2. *The Prym-Tyurin varieties are examples of abelian varieties which are embedded in Jacobians [BL][section 12.2]. For them the above result is true. Recall that, an abelian variety (A, Ξ) is called a Prym-Tyurin variety if there exists a smooth projective curve C such that A is embedded in $J(C)$ and we have the following property. Let Θ be the theta divisor of $J(C)$, and i denote the embedding of A into $J(C)$, then*

$$i^*(\Theta) \equiv e\Xi$$

for some integer e . By theorem 2.1 it follows that j_ from $CH_*(i^*\Theta)$ to $CH_*(A)$ is injective, where j denote the embedding of $i^*\Theta$ into A .*

Any principally polarized abelian variety is a Prym-Tyurin variety of a certain exponent e , see corollary 12.2.4 [BL]. Therefore by 2.1 we have the following result.

Theorem 2.3. *Let A be a principally polarized abelian variety embedded into $J(C)$ for some smooth projective curve C . Let Θ be the theta divisor of $J(C)$. Then the embedding of $\Theta \cap A$ into A induces an injective push-forward homomorphism at the level of Chow groups with \mathbb{Q} -coefficients.*

Proof. It follows from 2.1 and corollary 12.2.4 in [BL]. \square

2.4. Special divisors coming from coverings of C . Consider a Galois covering \tilde{C} of C of degree n and branched along r points. Then by [BL] corollary 11.4.4 we have an embedding of $J(C) \rightarrow J(\tilde{C})$. So we have A embedded in $J(\tilde{C})$. Consider $\Theta_{\tilde{C}} \cap A = \Theta_A$. Then we can prove that the embedding of Θ_A in A induces an injective push-forward homomorphism at the level of Chow groups with \mathbb{Q} -coefficients.

Theorem 2.5. *Let j_A denote the embedding of Θ_A in A . Then the push-forward homomorphism j_{A*} is injective at the level of Chow groups with \mathbb{Q} coefficients.*

Proof. The proof goes same as in theorem 4.1 in [BI] with the role of $J(C)$ replaced by A and that of Θ_C replaced by Θ_A . \square

2.6. An application of the previous result. Here we prove that the divisor Θ_A gives rise to a divisor on the Kummers quartic K3 surface associated to a principally polarized abelian surface A , such that the closed embedding of the divisor into the K3 surface induces injection at the level of Chow groups of zero cycles.

To prove that first we show that the embedding of Θ_C into $J(C)$ gives rise to an injection at the level of Chow groups. Although this has been proved in [BI][theorem 3.1], here we present an alternative proof following [Co] which gives a better understanding of the picture when we blow up $J(C)$ along some subvariety (in our case finitely many points).

Theorem 2.7. *The embedding of the theta divisor Θ_C into $J(C)$ for a smooth projective curve C , gives rise to an injection at the level of Chow groups.*

Proof. We use the fact that $\text{Sym}^g C$ maps surjectively and birationally onto $J(C)$ and $\text{Sym}^{g-1} C$ maps surjectively and birationally onto Θ_C . We have a natural correspondence Γ given by $\pi_g \times \pi_{g-1}(\text{Graph}(pr))$, where pr is the projection from C^g to C^{g-1} , π_i is the natural morphism from C^i to $\text{Sym}^i C$. Consider the correspondence Γ_1 on $J(C) \times \Theta_C$ given by $(f_1 \times f_2)(\Gamma)$, where f_1, f_2 are natural morphisms from $\text{Sym}^{g-1} C, \text{Sym}^g C$ to $\Theta_C, J(C)$ respectively. Then by projection formula it follows that

$$\Gamma_{1*} j_*$$

is induced by $(j \times id)^*(\Gamma_1)$ where j is the closed embedding of Θ_C into $J(C)$. Now we compute the cycle

$$(j \times id)^*(\Gamma_1)$$

that is nothing but the collection of divisors

$$(D_1, D_2)$$

such that D_1 is linearly equivalent to $\sum_{i=1}^{g-1} x_i - (g-1)p$ and D_2 is linearly equivalent to $\sum_{i=1}^{g-1} y_i - (g-1)p$, where

$$([x_1, \dots, x_{g-1}, p], [y_1, \dots, y_{g-1}]) \in \Gamma .$$

Therefore without loss of generality we can assume that elements in $(j \times id)^*(\Gamma_1)$ are classes of effective divisors on C of the form

$$([x_1 + \cdots + x_{g-1} + p - gp], [y_1 + \cdots + y_{g-1} + p - gp])$$

such that

$$([x_1, \cdots, x_{g-1}, p], [y_1, \cdots, y_{g-1}]) \in \Gamma$$

so we get that either

$$x_i = y_i$$

for all i or

$$y_i = p$$

for some i . Therefore we get that $(j \times id)^*\Gamma_1$ is equal to

$$\Delta + Y$$

where Y is supported on $\Theta_C \times \sum_{i=1}^{g-2} C_i$. The fact that the multiplicity of Δ in $(j \times id)^*(\Gamma_1)$ is 1 follows from the fact that $\text{Sym}^g C$ maps surjectively and birationally onto $J(C)$, and the computations following [Co]. Also here the Chow moving lemma holds for $\Theta_C \times \Theta_C$, because it holds for $\text{Sym}^{g-1} C \times \text{Sym}^{g-1} C$ with the cycles taken with \mathbb{Q} -coefficients and the fact that f_1 is birational. So let

$$\rho : U \rightarrow \Theta_C$$

be the open embedding of the complement of $\sum_{i=1}^{g-2} C_i$ in Θ_C . Then we have

$$\rho^* \Gamma_{1*} j_*(Z) = \rho^*(Z + Z_1) = \rho^*(Z)$$

where Z_1 is supported on $\sum_{i=1}^{g-2} C_i$. This follows since $(j \times id)^*(\Gamma_1) = \Delta + Y$, where Y is supported on $\Theta_C \times \sum_{i=1}^{g-2} C_i$. Now consider the following commutative diagram.

$$\begin{array}{ccccc} CH_*(\sum_{i=1}^{g-2} C_i) & \xrightarrow{j'_*} & CH_*(\Theta_C) & \xrightarrow{\rho^*} & CH_*(U) \\ \downarrow & & \downarrow j_* & & \downarrow \\ CH_*(\sum_{i=1}^{g-2} C_i) & \xrightarrow{j''_*} & CH_*(J(C)) & \longrightarrow & CH^*(V) \end{array}$$

Here U, V are complements of $\sum_{i=1}^{g-2} C_i$ in $\Theta_C, J(C)$ respectively. Now suppose that $j_*(z) = 0$, then from the previous it follows that

$$\rho^* \Gamma_{1*} j_*(z) = \rho^*(z) = 0$$

by the localisation exact sequence it follows that there exists z' in $\sum_{i=1}^{g-2} C_i$ such that $j'_*(z') = z$. By the commutativity and the induction hypothesis it follows that

$$j''_*(z') = 0$$

since $\sum_{i=1}^{g-2} C_i$ is of dimension $g-2$. So we get that $z' = 0$ hence $z = 0$. So j_* is injective. \square

Now we prove that if we blow up $J(C)$ at finitely many points and denote the blow up by $\widetilde{J(C)}$ and let $\widetilde{\Theta}_C$ denote the total transform of Θ_C , then the closed embedding of $\widetilde{\Theta}_C$ into $\widetilde{J(C)}$ induces injective push-forward homomorphism at the level of Chow groups.

Theorem 2.8. *Let $\widetilde{J(C)}$ be the blow up of $J(C)$ at some non-singular subvariety Z whose inverse image is E . Let Θ_C intersect Z transversely. Let $\widetilde{\Theta}_C$ denote the strict transform of Θ_C in $\widetilde{J(C)}$. Then the closed embedding of $\widetilde{\Theta}_C$ into $\widetilde{J(C)}$ induces injective push-forward homomorphism at the level of Chow groups of zero cycles.*

Proof. Let us consider π to be the morphism from $\widetilde{J(C)}$ to $J(C)$. Consider the correspondence $(\pi' \times \pi)^*(\Gamma_1)$, where π' is the restriction of π to $\widetilde{\Theta}_C$. Call this correspondence Γ' . Then $\Gamma'_* \widetilde{j}_*$ is induced by $(\widetilde{j} \times id)^* \Gamma'$, where \widetilde{j} is the closed embedding of $\widetilde{\Theta}_C$ into $\widetilde{J(C)}$. Consider the commutative square.

$$\begin{array}{ccc} \widetilde{\Theta}_C \times \widetilde{\Theta}_C & \xrightarrow{\widetilde{j} \times id} & \widetilde{J(C)} \times \widetilde{J(C)} \\ \pi' \times \pi' \downarrow & & \downarrow \pi \times \pi \\ \Theta_C \times \Theta_C & \xrightarrow{j \times id} & J(C) \times J(C) \end{array}$$

This gives us that

$$(\widetilde{j} \times id)^* \Gamma' = (\pi' \times \pi')^*(j \times id)^* \Gamma_1 = (\pi' \times \pi')^*(\Delta + Y)$$

where Y is supported on $\Theta_C \times \sum_{i=1}^{g-2} C_i$. Now

$$(\pi' \times \pi')^*(\Delta) = \Delta + V$$

where E is the exceptional locus of π and V is supported on $(E \cap \widetilde{\Theta}_C) \times (E \cap \widetilde{\Theta}_C)$. So considering ρ to be the inclusion of the complement of

$\widetilde{\sum_{i=1}^{g-2} C_i}$ in $\widetilde{\Theta}_C$ and applying Chow moving lemma we have

$$\rho^* \Gamma'_* \tilde{j}_* = \rho^* .$$

Consider the following commutative diagram.

$$\begin{array}{ccccc} CH_0(A) & \xrightarrow{\tilde{j}'_*} & CH_0(\widetilde{\Theta}_C) & \xrightarrow{\rho_0^*} & CH_0(U) \\ \downarrow & & \downarrow \tilde{j}_* & & \downarrow \\ CH_0(A) & \xrightarrow{\tilde{j}''_*} & CH_0(\widetilde{J(C)}) & \longrightarrow & CH_0(V) \end{array}$$

Here $A = \widetilde{\sum_{i=1}^{g-2} C_i}$. Now suppose that $\tilde{j}_*(z) = 0$. By the previous computation we get that $\rho^*(z) = 0$, so by the localisation exact sequence we get that there exists z' in $CH_*(A)$ such that $\tilde{j}'_*(z') = z$.

By induction $CH_*(A) \rightarrow CH_*(\widetilde{J(C)})$ is injective. So we get that $z' = 0$ hence $z = 0$ giving \tilde{j}_* injective. □

Similar technique as in [B][proposition 3.1] tells us that if we consider an abelian variety A embedded in $J(C)$ and consider Θ_A to be $\Theta_C \cap A$, then the closed embedding of Θ_A into A induces injective push-forward homomorphism at the level of Chow groups. To be precise, we consider the correspondence Γ' to be restriction of Γ_1 to $A \times \Theta_A$ and proceed as in the previous theorem 2.8. Then that will give us $CH_*(\Theta_A)$ to $CH_*(A)$ is an injection and following the line of argument as in 2.8 we get that for a blow up of A , we have $CH_0(\widetilde{\Theta}_A)$ to $CH_0(\widetilde{A})$ is injective, where $\widetilde{\Theta}_A$ denote the total transform of Θ_A in the blow up \widetilde{A} .

Now let A be an abelian surface which is embedded in some $J(C)$. Let i denote the involution of A . Then i has 16 fixed points. We blow up A along these fixed points. Then we get \widetilde{A} on which we have an induced involution, call it i . Let $\widetilde{\Theta}_A$ denote the total transform of Θ_A in \widetilde{A} . Then the above discussion tells us the following.

Theorem 2.9. *The closed embedding of $\widetilde{\Theta}_A$ into \widetilde{A} induces injective push-forward homomorphism at the level of Chow groups of zero cycles.*

Now \tilde{A}/i is the Kummer's K3 surface associated to A . Suppose that we choose Θ_C such that it is i invariant. Then $\widetilde{\Theta_A}$ will be i invariant. The above theorem gives us:

Theorem 2.10. *The closed embedding of $\widetilde{\Theta_A}/i$ into \tilde{A}/i induces injective push-forward homomorphism at the level of Chow groups of zero cycles with \mathbb{Q} -coefficients.*

Note that all these techniques can be repeated if we consider the group of algebraic cycles modulo algebraic equivalence. Then therefore the closed embedding $\widetilde{\Theta_A}/i$ into \tilde{A}/i induces injection at the level of zero cycles modulo algebraic equivalence.

3. ABELIAN VARIETIES ISOGENOUS TO JACOBIAN VARIETIES

In this section we consider the abelian varieties which are isogenous to Jacobian varieties. By an isogeny we mean a surjective morphism of abelian varieties with finite kernel. Let f be an isogeny from $J(C)$ to an abelian variety A , where C is a smooth projective curve, such that $f(\Theta_C) = \Theta_A$ and assume that the restriction of f to Θ_C is finite. Here Θ_C, Θ_A denote the theta divisors in $J(C), A$. With this assumptions in hand we prove the following.

Theorem 3.1. *The closed embedding of Θ_A into A induces an injective push-forward homomorphism at the level of Chow groups with \mathbb{Q} coefficients.*

Proof. To prove the assertion let us consider the following Cartesian square.

$$\begin{array}{ccc} \Theta_C & \xrightarrow{f'} & \Theta_A \\ j_C \downarrow & & \downarrow j_A \\ J(C) & \xrightarrow{f} & A. \end{array}$$

Here the morphism f is flat as it is finite and surjective between two non-singular varieties and j_A is proper as it is a closed immersion. So by the push-forward pull-back formulae on Chow groups [[Fu]proposition

1.7] we get the following commutative diagram at the level of Chow groups.

$$\begin{array}{ccc}
CH_k(\Theta_A)_{\mathbb{Q}} & \xrightarrow{f'^*} & CH_k(\Theta_C)_{\mathbb{Q}} \\
j_{A*} \downarrow & & \downarrow j_{C*} \\
CH_k(A)_{\mathbb{Q}} & \xrightarrow{f^*} & CH_k(J(C))_{\mathbb{Q}}
\end{array}$$

That is we have

$$j_{C*}f'^* = f^*j_{A*}.$$

Now by repeating the arguments in [BI] theorem 3.1 we can prove that j_{C*} is injective at the level of Chow groups with \mathbb{Q} coefficients. The morphism f' is finite. So by example 1.7.4 in [Fu] we get that

$$f'_*f'^* = nId$$

where n is the degree of the finite map f' . Since we are working with \mathbb{Q} coefficients the above formula gives us that $f'_*f'^*$ is injective, hence f'^* is injective. So we get that f^*j_{A*} is injective, therefore j_{A*} is injective. \square

4. ABELIAN SCHEME EMBEDDED IN A FAMILY OF JACOBIANS

Let us consider a family of abelian varieties $\mathcal{A} \rightarrow S$ that is an abelian scheme such that \mathcal{A} is embedded in \mathcal{J} , where \mathcal{J} is a family of Jacobians of curves associated to the family of smooth projective curves $\mathcal{C} \rightarrow S$ and $\mathcal{J} \rightarrow S$ is a fibration. Let us consider the family of Theta divisors in \mathcal{J} to $\Theta_{\mathcal{J}}$, and consider $\Theta_{\mathcal{A}} := \Theta_{\mathcal{J}} \cap \mathcal{A}$. Let j denote the closed embedding of $\Theta_{\mathcal{A}}$ into \mathcal{A} . Then we prove that j_* from $CH_*(\Theta_{\mathcal{A}})$ to $CH_*(\mathcal{A})$ is injective.

Theorem 4.1. *The homomorphism j_* at the level of Chow groups is injective.*

Proof. To prove this theorem we use the localisation exact sequence for Higher Chow groups as present in [Bl]. Let us consider inverse image of $\mathcal{A}, \Theta_{\mathcal{A}}$ in $\text{Sym}^g \mathcal{C}, \text{Sym}^{g-1} \mathcal{C}$ respectively, denote them by $\mathcal{A}', \Theta'_{\mathcal{A}}$. Then we have the following commutative diagram with the rows exact.

$$\begin{array}{ccccccc}
CH_k(\mathrm{Sym}^{g-1}\mathcal{C}, 1) & \longrightarrow & CH_k(\mathrm{Sym}^{g-1}\mathcal{C} \setminus \Theta'_{\mathcal{A}}, 1) & \longrightarrow & CH_k(\Theta'_{\mathcal{A}}) & \longrightarrow & CH_k(\mathrm{Sym}^{g-1}\mathcal{C}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
CH_k(\mathrm{Sym}^g\mathcal{C}, 1) & \longrightarrow & CH_k(\mathrm{Sym}^g\mathcal{C} \setminus \mathcal{A}', 1) & \longrightarrow & CH_k(\mathcal{A}') & \longrightarrow & CH_k(\mathrm{Sym}^g\mathcal{C})
\end{array}$$

Now by an easy generalisation of Collino's theorem for higher Chow groups as proved in [BI] we have the push-forward homomorphism

$$CH_k(\mathrm{Sym}^{g-1}\mathcal{C}, 1) \rightarrow CH_k(\mathrm{Sym}^g\mathcal{C}, 1)$$

is injective and so is

$$CH_k(\mathrm{Sym}^{g-1}\mathcal{C} \setminus \Theta'_{\mathcal{A}}, 1) \rightarrow CH_k(\mathrm{Sym}^g\mathcal{C} \setminus \mathcal{A}')$$

We can generalise the proofs of [BI] section 5 for higher dimensional varieties. So we have $CH_k(\mathrm{Sym}^{g-1}\mathcal{C}) \rightarrow CH_k(\mathrm{Sym}^g\mathcal{C})$ injective. Now we want to prove that $CH_k(\Theta'_{\mathcal{A}}) \rightarrow CH_k(\mathcal{A}')$ is injective. Suppose we take some non-zero element in $CH_k(\Theta'_{\mathcal{A}})$. Suppose it is mapped to a non-zero element in $CH_k(\mathrm{Sym}^{g-1}\mathcal{C})$, then since $CH_k(\mathrm{Sym}^{g-1}\mathcal{C}) \rightarrow CH_k(\mathrm{Sym}^g\mathcal{C})$ is injective, we have that the non-zero element is mapped to a non-zero element in $CH_k(\mathrm{Sym}^g\mathcal{C})$. Therefore by the commutativity of the square

$$\begin{array}{ccc}
CH_k(\Theta'_{\mathcal{A}}) & \longrightarrow & CH_k(\mathrm{Sym}^{g-1}\mathcal{C}) \\
\downarrow & & \downarrow \\
CH_k(\mathcal{A}') & \longrightarrow & CH_k(\mathrm{Sym}^g\mathcal{C})
\end{array}$$

we obtain that the element we started with is mapped to a non-zero element in $CH_k(\mathcal{A}')$, under the homomorphism $CH_k(\Theta'_{\mathcal{A}}) \rightarrow CH_k(\mathcal{A}')$. Suppose that the element is mapped to zero in $CH_k(\mathrm{Sym}^{g-1}\mathcal{C})$. Then by the exactness of the first row we get that it is in the image of the homomorphism $CH_*(\mathrm{Sym}^{g-1}\mathcal{C} \setminus \Theta'_{\mathcal{A}}, 1) \rightarrow CH_k(\Theta'_{\mathcal{A}})$. Since the element we started with was non-zero, its pre-image in $CH_*(\mathrm{Sym}^{g-1}\mathcal{C} \setminus \Theta'_{\mathcal{A}}, 1)$ is non-zero. The homomorphism $CH_*(\mathrm{Sym}^{g-1}\mathcal{C} \setminus \Theta'_{\mathcal{A}}, 1) \rightarrow CH_*(\mathrm{Sym}^g\mathcal{C} \setminus \mathcal{A}', 1)$ is injective. So the pre-images of the element chosen are mapped

to non-zero elements in $CH_*(\text{Sym}^g \mathcal{C} \setminus \mathcal{A}')$. Now these elements can be mapped to zero in $CH_k(\mathcal{A}')$. If they are mapped to a non-zero element, we get that the element we started with is mapped to a non-zero element. If not we have the pre-images of the element are mapped inside the image of the homomorphism

$$CH_*(\text{Sym}^g \mathcal{C}, 1) \rightarrow CH_*(\text{Sym}^g \mathcal{C} \setminus \mathcal{A}', 1) ,$$

and these elements are supported on $\text{Sym}^{g-1} \mathcal{C} \setminus \Theta'_{\mathcal{A}'}$, we get that the element in $CH_k(\text{Sym}^g \mathcal{C})$ is in the image of the homomorphism $CH_k(\text{Sym}^{g-1} \mathcal{C}) \rightarrow CH_k(\text{Sym}^g \mathcal{C})$. Therefore we get an element in $CH_k(\text{Sym}^{g-1} \mathcal{C})$ which is mapped to the element that we started with in $CH_k(\Theta'_{\mathcal{A}'})$. Then by the localisation exact sequence it will follow that the element we started with is zero contradicting our assumption that it is non-zero. So the pre-images of the element we started with are mapped to a non-zero element in $CH_k(\mathcal{A}')$. So the push-forward homomorphism from $CH_k(\Theta'_{\mathcal{A}'})$ to $CH_k(\mathcal{A}')$ is injective. Now consider the following commutative diagram.

$$\begin{array}{ccc} CH_k(\Theta'_{\mathcal{A}'}) & \longrightarrow & CH_k(\mathcal{A}') \\ \downarrow & & \downarrow \\ CH_k(\Theta_{\mathcal{A}'}) & \longrightarrow & CH_k(\mathcal{A}) \end{array}$$

We have $\Theta'_{\mathcal{A}'} \rightarrow \Theta_{\mathcal{A}'}$ surjective and birational and $\mathcal{A}' \rightarrow \mathcal{A}$ surjective and birational. Now we can apply arguments as in theorem 3.1 in [BI] to deduce that the push-forward homomorphism from $CH_k(\Theta_{\mathcal{A}'})$ to $CH_k(\mathcal{A})$ is injective. \square

5. ABELIAN SCHEMES ISOGENOUS TO A FAMILY OF JACOBIANS

Let $\mathcal{A} \rightarrow S$ be a family of abelian varieties such that there exists a finite surjective morphism from $\mathcal{J} \rightarrow \mathcal{A}$, which gives a fiberwise isogeny. Also suppose that $\Theta_{\mathcal{J}}$ is mapped to $\Theta_{\mathcal{A}}$, that is the theta divisors of the family of Jacobian varieties \mathcal{J} are mapped to theta divisors in the family \mathcal{A} and this morphism is also finite. Now we prove the following.

Theorem 5.1. *Let f denote the morphism $\mathcal{J} \rightarrow \mathcal{A}$. Then we prove that the closed embedding $j : \Theta_{\mathcal{A}} \rightarrow \mathcal{A}$ induces an injection at the level of Chow groups.*

Proof. To prove the assertion let us consider the following Cartesian square.

$$\begin{array}{ccc} \Theta_{\mathcal{J}} & \xrightarrow{f'} & \Theta_{\mathcal{A}} \\ j' \downarrow & & \downarrow j \\ \mathcal{J} & \xrightarrow{f} & \mathcal{A}. \end{array}$$

Here the morphism f is flat as it is finite and surjective between two non-singular varieties and j is proper as it is a closed immersion. So by the push-forward pull-back formulae on Chow groups [[Fu]proposition 1.7] we get the following commutative diagram at the level of Chow groups.

$$\begin{array}{ccc} CH_k(\Theta_{\mathcal{A}})_{\mathbb{Q}} & \xrightarrow{f'^*} & CH_k(\Theta_{\mathcal{J}})_{\mathbb{Q}} \\ j_* \downarrow & & \downarrow j'_* \\ CH_k(\mathcal{A})_{\mathbb{Q}} & \xrightarrow{f^*} & CH_k(\mathcal{J})_{\mathbb{Q}} \end{array}$$

That is we have

$$j'_* f'^* = f^* j_* .$$

Now by repeating the arguments in [BI] theorem 3.1 we can prove that j'_* is injective at the level of Chow groups with \mathbb{Q} coefficients. The morphism f' is finite. So by example 1.7.4 in [Fu] we get that

$$f'_* f'^* = n Id$$

where n is the degree of the finite map f' . Since we are working with \mathbb{Q} coefficients the above formula gives us that $f'_* f'^*$ is injective, hence f'^* is injective. So we get that $f^* j_*$ is injective, therefore j_* is injective. \square

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